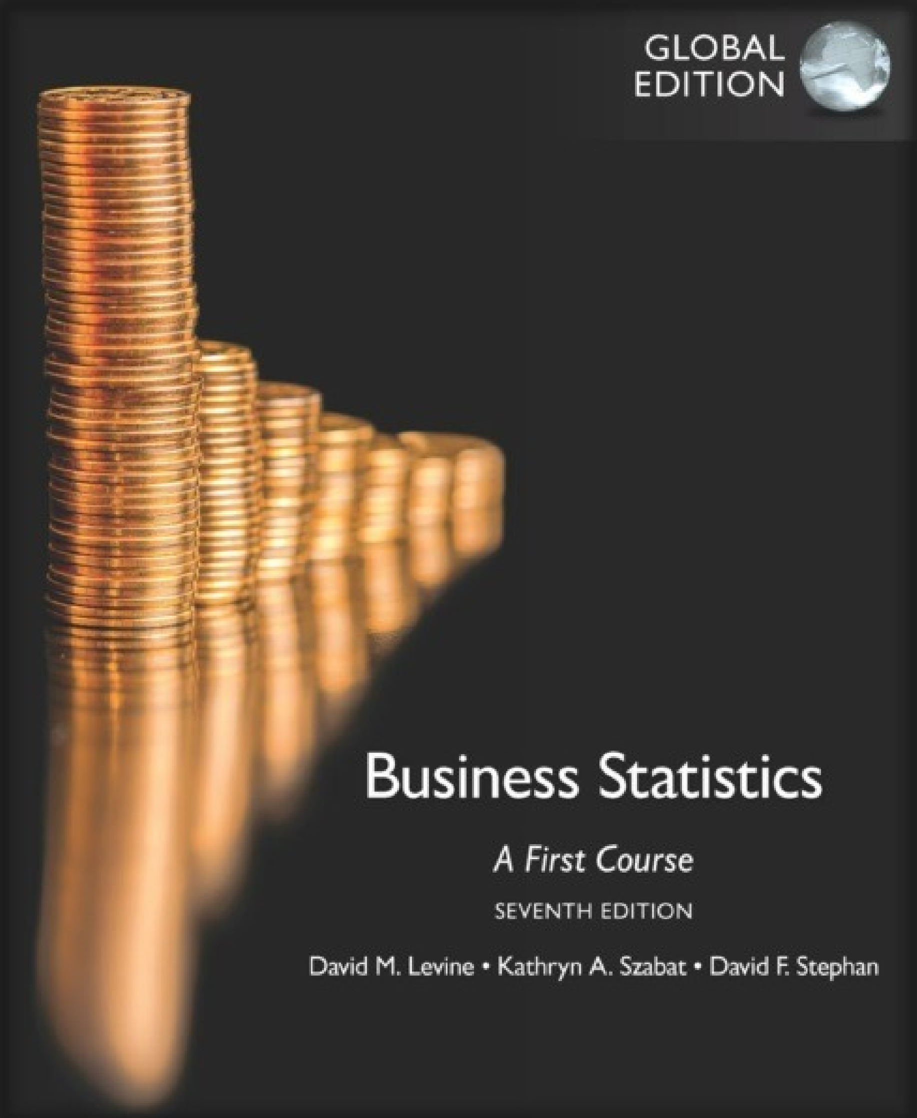


Chapter 5

Discrete Probability Distributions

Objectives

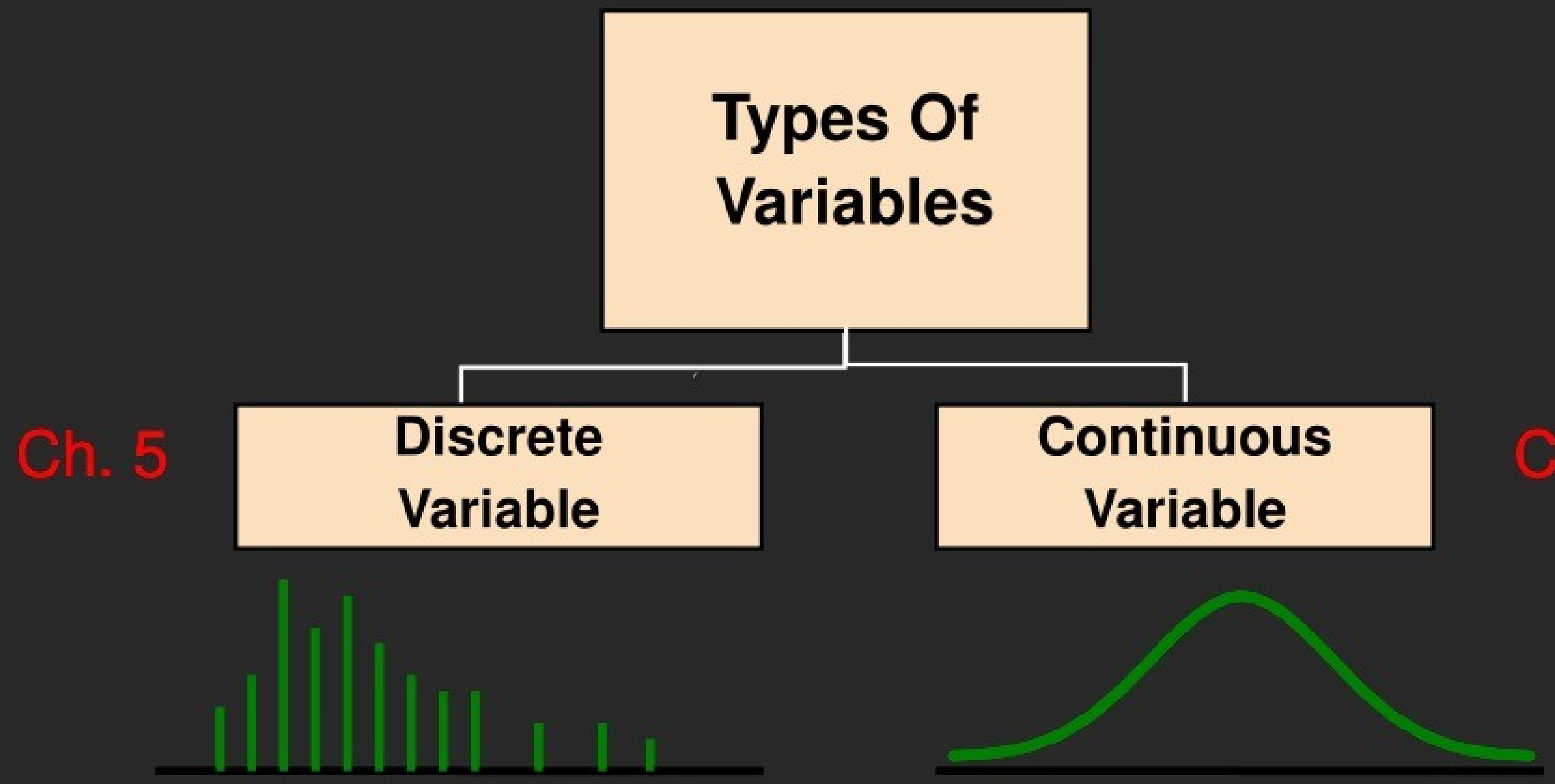
In this chapter, you learn:

- The properties of a probability distribution.
- To compute the expected value and variance of a probability distribution.
- To compute probabilities from binomial, and Poisson distributions.
- To use the binomial, and Poisson distributions to solve business problems

Definitions

- **Discrete** variables produce outcomes that come from a counting process (e.g. number of classes you are taking).
- **Continuous** variables produce outcomes that come from a measurement (e.g. your annual salary, or your weight).

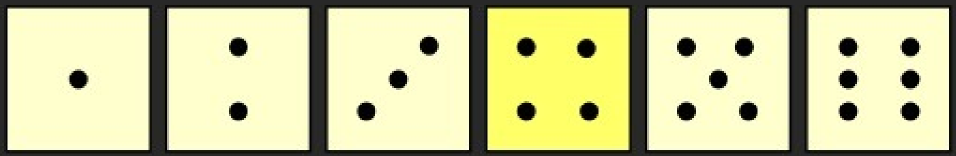
Types Of Variables



Discrete Variables

- Can only assume a countable number of values

Examples:



- **Roll a die twice**
  Let X be the number of times 4 occurs
  (then X could be 0, 1, or 2 times)

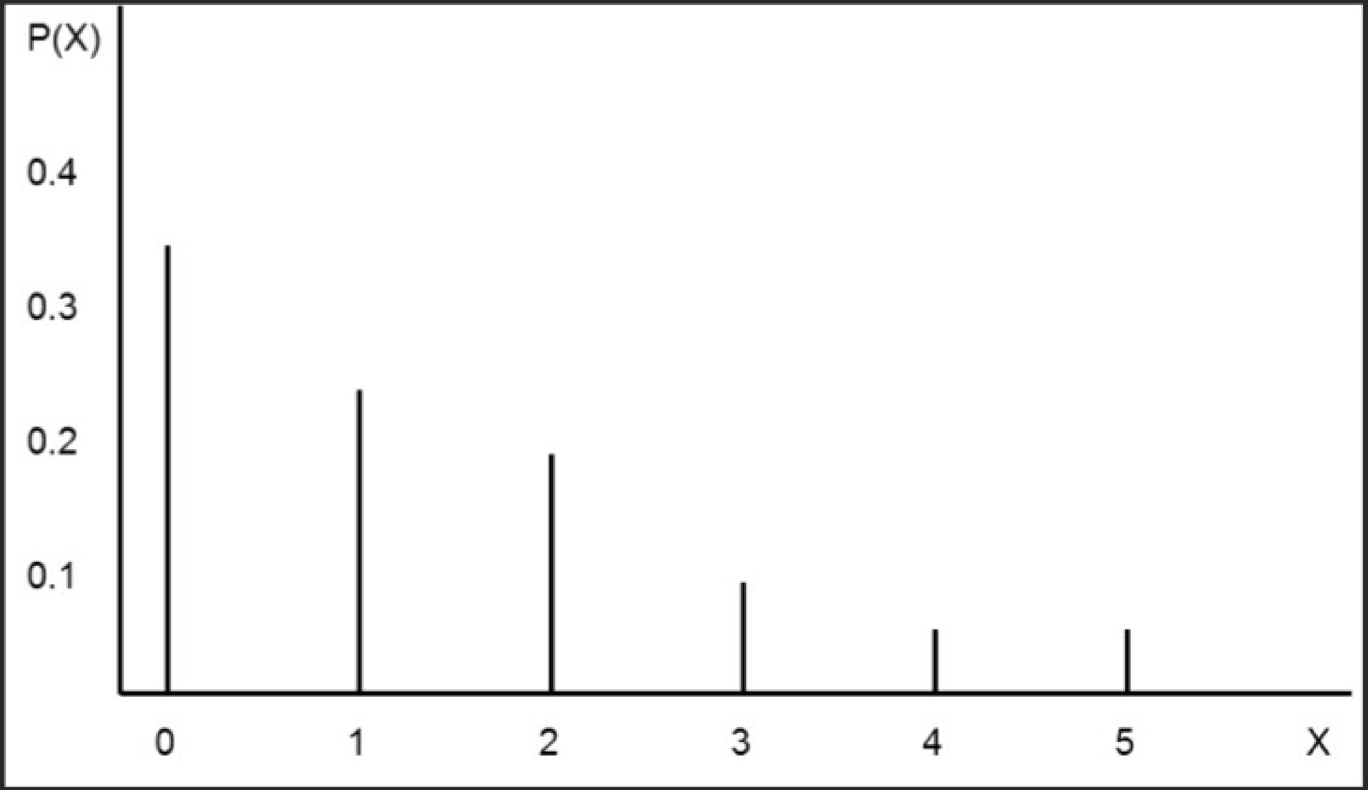
- **Toss a coin 5 times.**
  Let X be the number of heads
  (then X = 0, 1, 2, 3, 4, or 5)

Probability Distribution For A Discrete Variable

- A **probability distribution for a discrete variable** is a mutually exclusive listing of all possible numerical outcomes for that variable and a probability of occurrence associated with each outcome.

Interruptions Per Day In Computer Network	Probability
0	0.35
1	0.25
2	0.20
3	0.10
4	0.05
5	0.05

Probability Distributions Are Often Represented Graphically



Discrete Variables Expected Value (Measuring Center)

- Expected Value (or mean) of a discrete variable (Weighted Average)

$$\mu = E(X) = \sum_{i=1}^{N} x_i P(X = x_i)$$

Interruptions Per Day In Computer Network (x_i)	Probability $P(X = x_i)$	$x_i P(X = x_i)$
0	0.35	(0)(0.35) = 0.00
1	0.25	(1)(0.25) = 0.25
2	0.20	(2)(0.20) = 0.40
3	0.10	(3)(0.10) = 0.30
4	0.05	(4)(0.05) = 0.20
5	0.05	(5)(0.05) = 0.25
	1.00	μ = E(X) = 1.40

Discrete Variables: Measuring Dispersion

(continued)

$$\sigma = \sqrt{\sum_{i=1}^{N} [x_i - E(X)]^2 P(X = x_i)}$$

Interruptions Per Day In Computer Network (x_i)	Probability $P(X = x_i)$	$[x_i - E(X)]^2$	$[x_i - E(X)]^2 P(X = x_i)$
0	0.35	$(0 - 1.4)^2 = 1.96$	$(1.96)(0.35) = 0.686$
1	0.25	$(1 - 1.4)^2 = 0.16$	$(0.16)(0.25) = 0.040$
2	0.20	$(2 - 1.4)^2 = 0.36$	$(0.36)(0.20) = 0.072$
3	0.10	$(3 - 1.4)^2 = 2.56$	$(2.56)(0.10) = 0.256$
4	0.05	$(4 - 1.4)^2 = 6.76$	$(6.76)(0.05) = 0.338$
5	0.05	$(5 - 1.4)^2 = 12.96$	$(12.96)(0.05) = 0.648$
			$\sigma^2 = 2.04$, $\sigma = 1.4283$

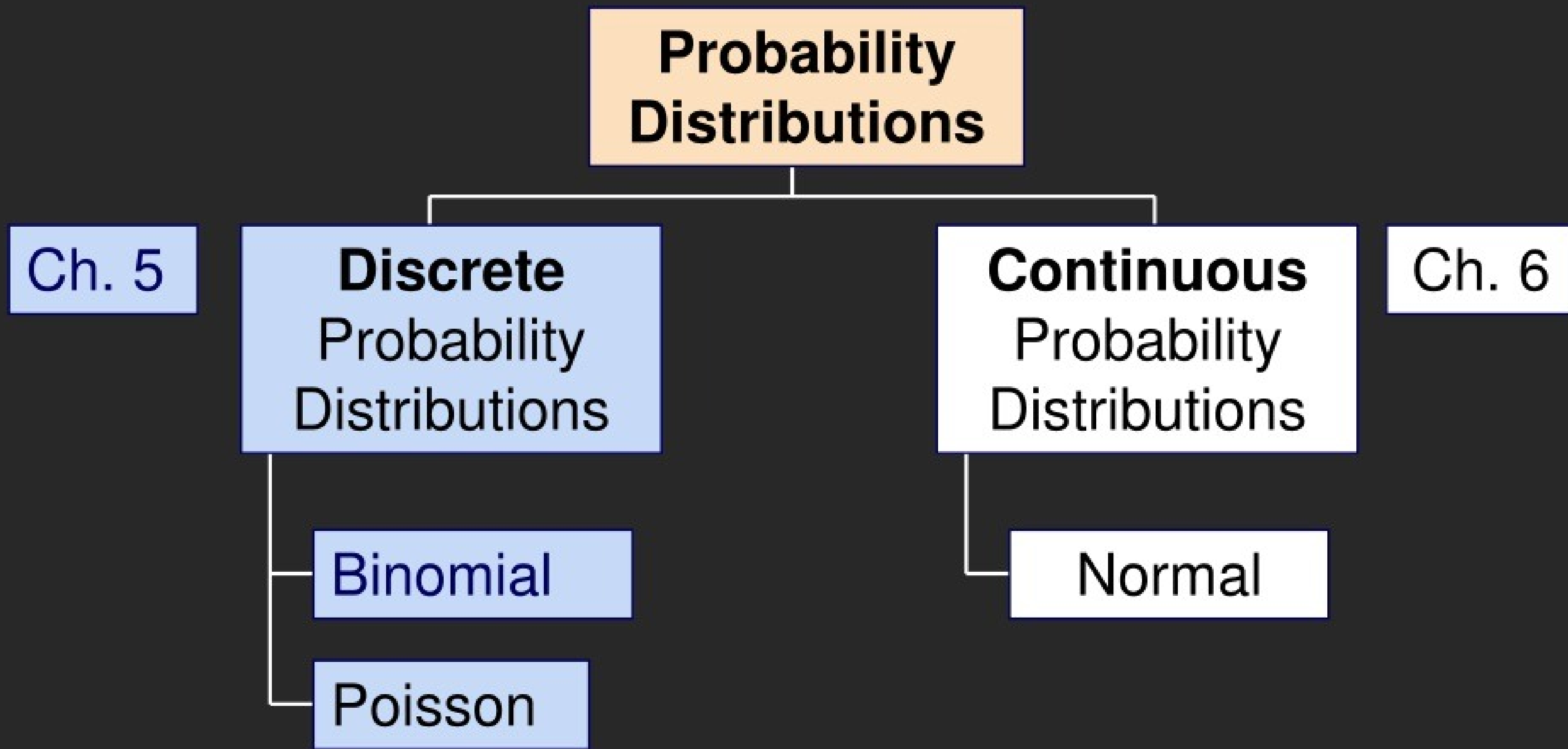
Discrete Variables: Measuring Dispersion

(continued)

$$\sigma = \sqrt{\sum_{i=1}^{N} [x_i - E(X)]^2 P(X = x_i)}$$

Interruptions Per Day In Computer Network (x_i)	Probability $P(X = x_i)$	$[x_i - E(X)]^2$	$[x_i - E(X)]^2 P(X = x_i)$
0	0.35	$(0 - 1.4)^2 = 1.96$	(1.96)(0.35) = 0.686
1	0.25	$(1 - 1.4)^2 = 0.16$	(0.16)(0.25) = 0.040
2	0.20	$(2 - 1.4)^2 = 0.36$	(0.36)(0.20) = 0.072
3	0.10	$(3 - 1.4)^2 = 2.56$	(2.56)(0.10) = 0.256
4	0.05	$(4 - 1.4)^2 = 6.76$	(6.76)(0.05) = 0.338
5	0.05	$(5 - 1.4)^2 = 12.96$	(12.96)(0.05) = 0.648
			$\sigma^2 = 2.04$, $\sigma = 1.4283$

Probability Distributions



Binomial Probability Distribution

(continued)

- Observations are independent
  - The outcome of one observation does not affect the outcome of the other
  - Two sampling methods deliver independence
    - Infinite population without replacement
    - Finite population with replacement

Binomial Probability Distribution

(continued)

- Observations are independent
  - The outcome of one observation does not affect the outcome of the other
  - Two sampling methods deliver independence
    - Infinite population without replacement
    - Finite population with replacement

Possible Applications for the Binomial Distribution

- A manufacturing plant labels items as either defective or acceptable
- A firm bidding for contracts will either get a contract or not
- A marketing research firm receives survey responses of “yes I will buy” or “no I will not”
- New job applicants either accept the offer or reject it

The Binomial Distribution Counting Techniques

- Suppose the event of interest is obtaining heads on the toss of a fair coin. You are to toss the coin three times. In how many ways can you get two heads?
- Possible ways: HHT, HTH, THH, so there are three ways you can getting two heads.
- This situation is fairly simple. We need to be able to count the number of ways for more complicated situations.

Counting Techniques
Rule of Combinations

- How many possible 3 scoop combinations could you create at an ice cream parlor if you have 31 flavors to select from and no flavor can be used more than once in the 3 scoops?
- The total choices is n = 31, and we select X = 3.

$$ {}_{31}C_3 = \frac{31!}{3!(31-3)!} = \frac{31!}{3!28!} = \frac{31 \cdot 30 \cdot 29 \cdot 28!}{3 \cdot 2 \cdot 1 \cdot 28!} = 31 \cdot 5 \cdot 29 = 4,495 $$

Counting Techniques
Rule of Combinations

- How many possible 3 scoop combinations could you create at an ice cream parlor if you have 31 flavors to select from and no flavor can be used more than once in the 3 scoops?
- The total choices is n = 31, and we select X = 3.

$$ {}_{31}C_3 = \frac{31!}{3!(31-3)!} = \frac{31!}{3!28!} = \frac{31 \times 30 \times 29 \times 28!}{3 \times 2 \times 1 \times 28!} = 31 \times 5 \times 29 = 4{,}495 $$

Binomial Distribution Formula

$$P(X{=}x\,|n,\pi) = \frac{n!}{x!\;(n-x)!}\,\pi^{x}\,(1-\pi)^{n-x}$$

$P(X{=}x|n,\pi)$ = probability of **x** events of interest in **n** trials, with the probability of an "event of interest" being **π** for each trial

x = number of "events of interest" in sample, (x = 0, 1, 2, ..., n)

n = sample size (number of trials or observations)

π = probability of "event of interest"

Example: Flip a coin four times, let x = # heads:

n = 4

π = 0.5

1 - π = (1 - 0.5) = 0.5

X = 0, 1, 2, 3, 4

Example: Calculating a Binomial Probability

What is the probability of one success in five observations if the probability of an event of interest is 0.1?

$x = 1$, $n = 5$, and $\pi = 0.1$

$$P(X = 1 | 5, 0.1) = \frac{n!}{x!(n-x)!}\pi^x(1-\pi)^{n-x}$$

$$= \frac{5!}{1!(5-1)!}(0.1)^1(1-0.1)^{5-1}$$

$$= (5)(0.1)(0.9)^4$$

$$= 0.32805$$

The Binomial Distribution Example

Suppose the probability of purchasing a defective computer is 0.02. What is the probability of purchasing 2 defective computers in a group of 10?

$x = 2$, $n = 10$, and $\pi = 0.02$

$$P(X = 2 \mid 10, 0.02) = \frac{n!}{x!(n-x)!}\pi^x(1-\pi)^{n-x}$$

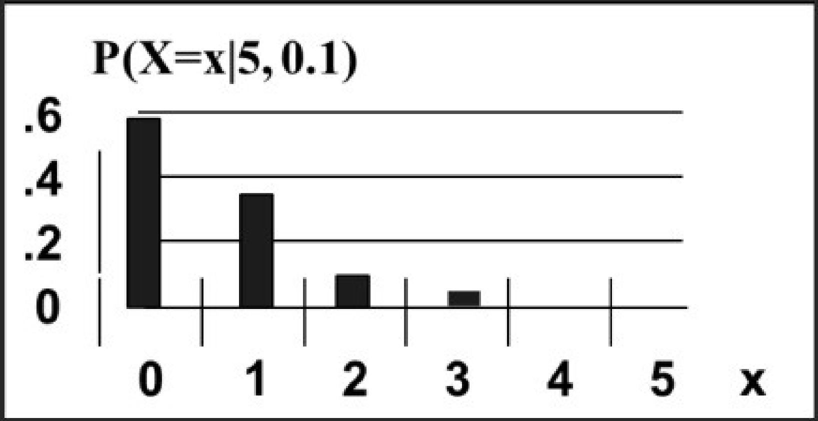
$$= \frac{10!}{2!(10-2)!}(.02)^2(1-.02)^{10-2}$$

$$= (45)(.0004)(.8508)$$

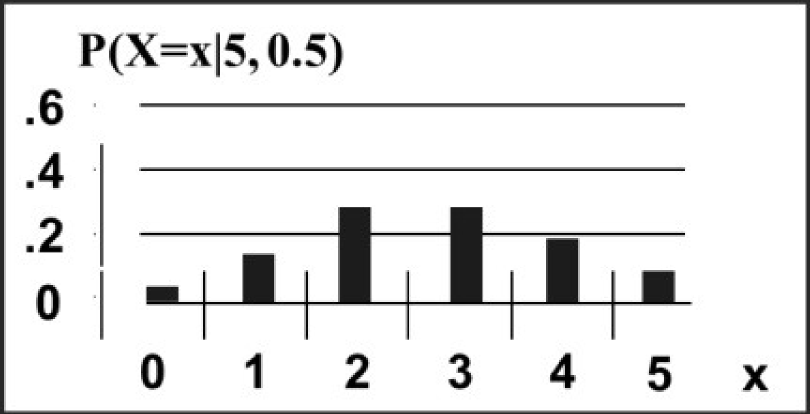
$$= .01531$$

The Binomial Distribution Shape

- The shape of the binomial distribution depends on the values of π and n
- Here, n = 5 and π = .1



- Here, n = 5 and π = .5



The Binomial Distribution Using Binomial Tables (Available On Line)

n = 10									
x	…	π=.20	π=.25	π=.30	**π=.35**	π=.40	π=.45	π=.50	
0	…	0.1074	0.0563	0.0282	0.0135	0.0060	0.0025	0.0010	10
1	…	0.2684	0.1877	0.1211	0.0725	0.0403	0.0207	0.0098	9
2	…	0.3020	**0.2816**	0.2335	0.1757	0.1209	0.0763	0.0439	8
3	…	0.2013	0.2503	0.2668	**0.2522**	0.2150	0.1665	0.1172	7
4	…	0.0881	0.1460	0.2001	0.2377	0.2508	0.2384	0.2051	6
5	…	0.0264	0.0584	0.1029	0.1536	0.2007	0.2340	0.2461	5
6	…	0.0055	0.0162	0.0368	0.0689	0.1115	0.1596	0.2051	4
7	…	0.0008	0.0031	0.0090	0.0212	0.0425	0.0746	0.1172	3
8	…	0.0001	0.0004	0.0014	0.0043	0.0106	0.0229	0.0439	**2**
9	…	0.0000	0.0000	0.0001	0.0005	0.0016	0.0042	0.0098	1
10	…	0.0000	0.0000	0.0000	0.0000	0.0001	0.0003	0.0010	0
	…	π=.80	**π=.75**	π=.70	π=.65	π=.60	π=.55	π=.50	**x**

Examples:

$n = 10, \pi = 0.35, x = 3$: $P(X = 3|10, 0.35) = 0.2522$

$n = 10, \pi = 0.75, x = 8$: $P(X = 8|10, 0.75) = 0.2816$

Binomial Distribution Characteristics

- Mean

$$\mu = E(X) = n\pi$$

- Variance and Standard Deviation

$$\sigma^2 = n\pi(1-\pi)$$

$$\sigma = \sqrt{n\pi(1-\pi)}$$

Where n = sample size

π = probability of the event of interest for any trial

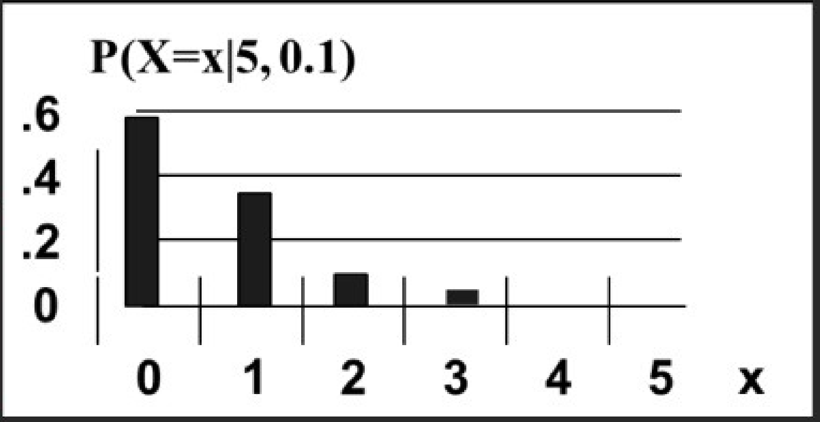
$(1 - \pi)$ = probability of no event of interest for any trial

The Binomial Distribution Characteristics

Examples

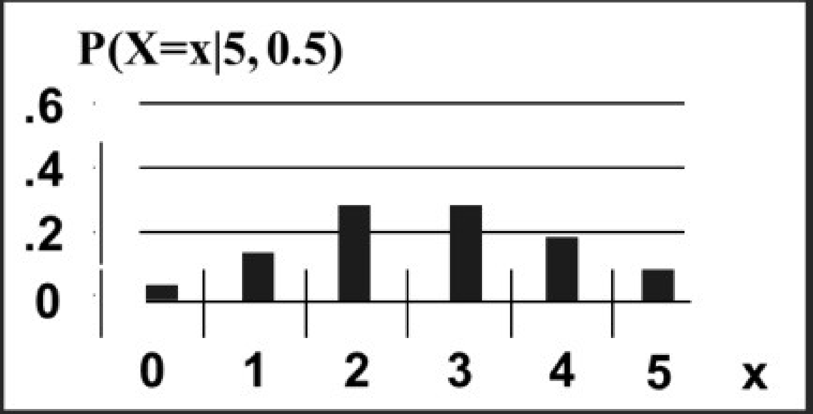
$$\mu = n\pi = (5)(.1) = 0.5$$

$$\sigma = \sqrt{n\pi(1-\pi)} = \sqrt{(5)(.1)(1-.1)} = 0.6708$$



$$\mu = n\pi = (5)(.5) = 2.5$$

$$\sigma = \sqrt{n\pi(1-\pi)} = \sqrt{(5)(.5)(1-.5)} = 1.118$$



The Poisson Distribution Definitions

- You use the **Poisson distribution** when you are interested in the number of times an event occurs in a given **area of opportunity**.
- An **area of opportunity** is a continuous unit or interval of time, volume, or such area in which more than one occurrence of an event can occur.
  - The number of scratches in a car's paint
  - The number of mosquito bites on a person
  - The number of computer crashes in a day

The Poisson Distribution

- Apply the Poisson Distribution when:
  - You wish to count the number of times an event occurs in a given area of opportunity
  - The probability that an event occurs in one area of opportunity is the same for all areas of opportunity
  - The number of events that occur in one area of opportunity is independent of the number of events that occur in the other areas of opportunity
  - The probability that two or more events occur in an area of opportunity approaches zero as the area of opportunity becomes smaller
  - The average number of events per unit is λ (lambda)

Poisson Distribution Formula

$$P(X = x \mid \lambda) = \frac{e^{-\lambda}\lambda^{x}}{x!}$$

where:

- x = number of events in an area of opportunity
- λ = expected number of events
- e = base of the natural logarithm system (2.71828...)

Poisson Distribution Characteristics

- Mean

$$\mu \bullet \lambda$$

- Variance and Standard Deviation

$$\sigma^2 \bullet \lambda$$

$$\sigma \bullet \sqrt{\lambda}$$

where ✦ = expected number of events

Using Poisson Tables (Available On Line)

X	λ								
	0.10	0.20	0.30	0.40	**0.50**	0.60	0.70	0.80	0.90
0	0.9048	0.8187	0.7408	0.6703	0.6065	0.5488	0.4966	0.4493	0.4066
1	0.0905	0.1637	0.2222	0.2681	0.3033	0.3293	0.3476	0.3595	0.3659
2	0.0045	0.0164	0.0333	0.0536	**0.0758**	0.0988	0.1217	0.1438	0.1647
3	0.0002	0.0011	0.0033	0.0072	0.0126	0.0198	0.0284	0.0383	0.0494
4	0.0000	0.0001	0.0003	0.0007	0.0016	0.0030	0.0050	0.0077	0.0111
5	0.0000	0.0000	0.0000	0.0001	0.0002	0.0004	0.0007	0.0012	0.0020
6	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0001	0.0002	0.0003
7	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

Example: Find $P(X = 2 \mid \lambda = 0.50)$

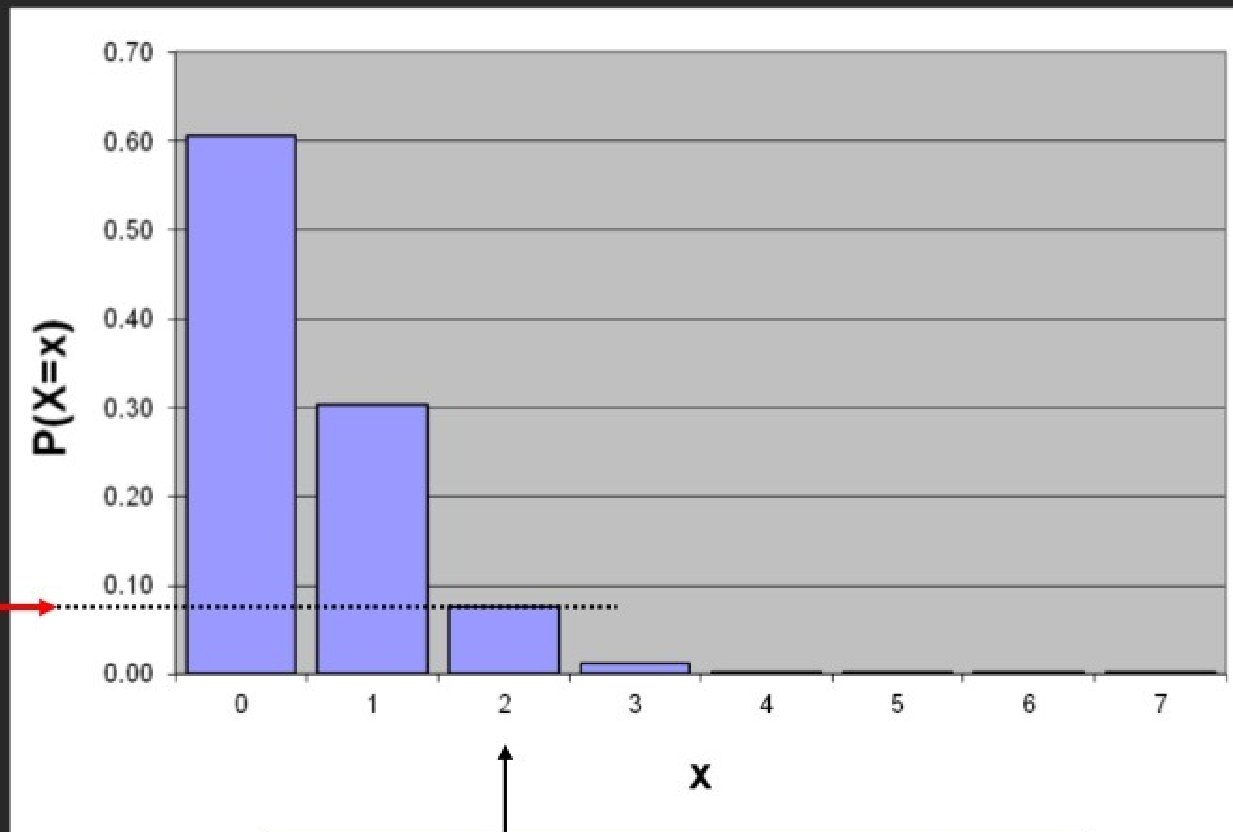
$$P(X = 2 \mid 0.50) = \frac{e^{-\lambda}\lambda^{x}}{x!} = \frac{e^{-0.50}(0.50)^{2}}{2!} = 0.0758$$

Graph of Poisson Probabilities

Graphically:

$\lambda = 0.50$

X	$\lambda =$ 0.50
0	0.6065
1	0.3033
2	0.0758
3	0.0126
4	0.0016
5	0.0002
6	0.0000
7	0.0000

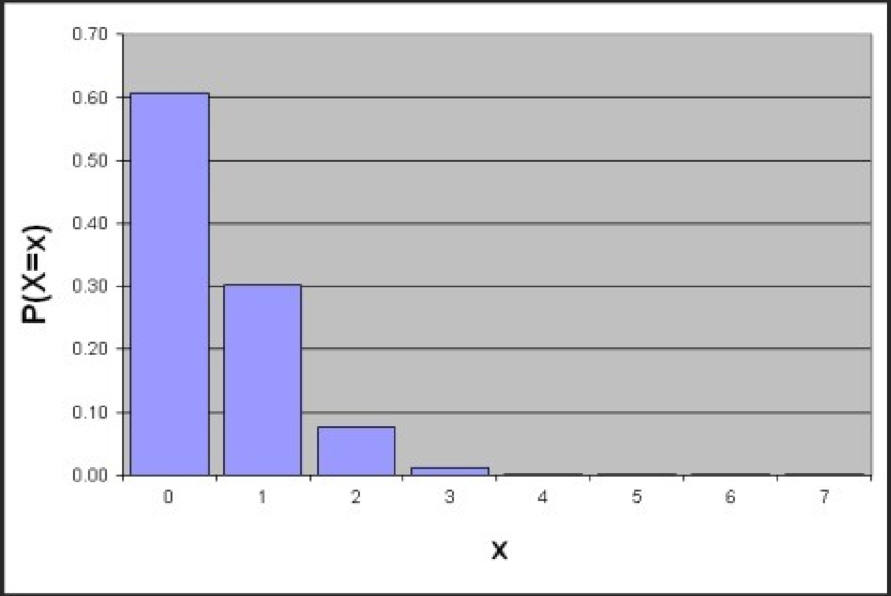


$P(X = 2 \mid \lambda = 0.50) = 0.0758$

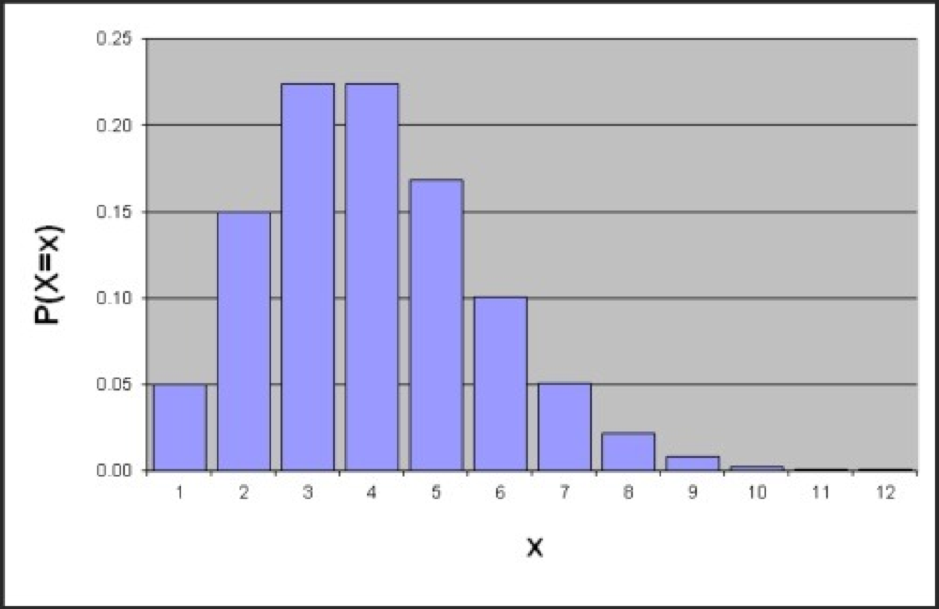
Poisson Distribution Shape

- The shape of the Poisson Distribution depends on the parameter λ :

λ = 0.50



λ = 3.00



Chapter Summary

In this chapter we covered:

- The properties of a probability distribution.
- To compute the expected value and variance of a probability distribution.
- To compute probabilities from binomial, and Poisson distributions.
- To use the binomial, and Poisson distributions to solve business problems